Military R&D, Growth and the Optimal Allocation of Government Spending

Stefano BOSI and Thierry LAURENT*
Center for Economic Policy Studies (EPEE)
Department of Economics - University of Paris-Evry

June 28, 2006

Abstract

The purpose of the model developed in the paper is to provide a simple economic framework to address an economic policy question, namely the optimal size of military R&D investment within total public expenditures. To capture the long run impact of military R&D on the growth rate of the economy, we develop an endogenous growth model in the line with Barro [1990] and Shieh & *alii* [2002]; the model focuses on the optimal sharing of public resources between civil investments, public consumption, military R&D investment and "standard" military spending. It emphasizes the key role played by public military R&D investments in determining the long run levels of economic growth and welfare. According to numerical simulations – based on a very prudential set of assumptions concerning the economic impact of military R&D – a 1.32 billions euros permanent reallocation of public spending from civilian unproductive public consumption toward military R&D investment, induces a GDP discounted benefit close to 100 billions euros on a decade. In such a framework, characterized by productive externalities originating in military R&D, the Government optimal economic policy should be to massively invest in military R&D: a global tax rate below 13%, would drive to a 8% GDP long run annual growth rate.

JEL: D58, H23, O31, O32.

Keywords: Endogenous Growth, R&D, Military Expenditures, Public Spending.

1 Introduction

Is the military spending a useless spending? A large number of empirical and theoretical papers investigate the economic effects of military spending on economic growth with no clear and definitive answer to this question. The so-called

*EPEE, Center for Economic Policy Studies, Department of Economics, University Paris-Evry, 4 bd. François Mitterrand, 91025 Evry cedex, France. Mail : laurent@univ-evry.fr, stefano.bosi@univ-evry.fr.

theory of *peace dividends* (McNamara [1991]), suggesting that during peace times milex wastes and diverts away important resources from the civil sector, is broadly put forward to explain why the milex impact on economic growth is frequently found to be non-significant or even negative (see for example Barro & Sala-i-Martin [1995]), or less efficient than civilian public spending (Macnair & *alii* [1995]).

The early cross-country analysis by Benoit [1973, 1978], concluding to a positive relationship between external threats and economic growth, opened the way to generations of empirical models reflecting different theoretical framework, econometric procedures or sampling (time series, cross-section etc.) and leading to widely various results. Lipow [1990], Macnair & *alii* [1995], Brumm [1997] and Murdoch & *alii* [1997] confirm Benoit [1973] conclusions while, for instance, Biswas & Ram [1986] as well as Huand & Mintz [1991] find no significant correlation between military public spending and growth. Deger & Smith [1983], Faini & *alii* [1984] and Deger [1986] even show the existence of a negative relationship between military expenditures and growth.[1]

Despite contrasted empirical results, the theoretical economic literature devoted to the analysis of the effects of military spending upon growth and welfare, has suggested three main channels through which milex can influence GDP. *Demand effects* originate in public spending multiplier consequences of milex: from a Keynesian point of view military spending promotes activity by stimulating the aggregate demand, and eventually, employment, growth and welfare through a standard multiplier mechanism.[2] *Supply effects* describe the impact of military spending on the efficiency of the production process: in the long run, positive externalities associated to military innovations, spill over the entire productive system, affecting both the quantity and productivity of inputs, which together determine potential output. *Safety effects* highlight the crucial role played by the National Defense in protecting persons and properties from domestic or foreign threats: safety conditions being an essential component of the incentives to invest and innovate, military expenditures, to the extent they increase national security, contribute to increase GDP (*cf.* for example Aizenman & Glick [2003]).

The purpose of this article is to study the long run impact, on economic growth and social welfare of a very specific component of milex, namely military R&D expenditures, and to define the optimal economic policy concerning the optimal share of R&D military spending into total public spending. Our strong feeling and starting point, is that both the economic literature and almost all the Governments have surprisingly underestimate the crucial role played by military R&D investments and their global effects on input productivity through knowledge externalities. To give some insights on this question, we build a

[1] See Sandler & Hartley [1995], Ram [1995] or Dunne & *alii* [2004] for some surveys of the relationship between milex and growth.

[2] Of course, the extent to which military expenditures can crowd out other forms of expenditure such as civilian investment depends on how the former is financed (Dakurah & *alii* [2001]). From a classical point of view, any spending is necessarily financed through current taxes, inflationary taxes, or future taxes (Ricardian equivalence) and substitutes private investments through a crowding-out effect.

general equilibrium endogenous growth model which allow us to distinguish between productive and unproductive defense expenditures – whose effects transit through very different channels – and which can be used to compute the benefits of a permanent reallocation of public spending from civilian unproductive public consumption toward military R&D investments.

To avoid any further misunderstandings let us now define what one denotes by productive *vs* unproductive expenditures. Expenditures, eventually leading to a production costs cut, through a classical *supply side* effect, are called *productive*; this definition is voluntarily wide to include public substructures investments (airports, roads, communication networks etc.), public R&D investments and education spending as well as public subsidies to private R&D etc. An *unproductive* spending does not denote expenditures with no effects on the economy, but rather expenditures generating *demand side* effects through a Keynesian multiplier mechanism. Following this distinction, military R&D is a productive expenditure, because it contributes to decrease both the production cost of the defense service and – through an innovation diffusion process (positive externalities) – the production cost of the civilian good; on the other hand, military wages constitute an unproductive spending, because raising the wage bill positively affects aggregate demand in a Keynesian way.

From a theoretical point of view, one knows that a Government intervention is necessary to implement the first or the second best, when the economic activity is characterized by the existence of externalities, defined as economic phenomena having welfare effects not fully accounted for in the price and market system. A major class of externalities is constituted by public goods, such as transportation or communication networks, which are useful to all firms but whose corresponding investments cannot be realized by any single firm; in such a situation, without a public intervention, these goods are under-provided or not provided at all. Knowledge constitutes another case of externality, which stands at the heart of the present paper; as it cannot be the subject of property rights, one can consider it as a public good. Innovations, originating in R&D investments of one particular company, benefit, sooner or later, to other firms of the same sector and, step by step, to the whole economy; such transmission mechanisms allow us to understand how scientific and technical externalities bypass the market, by switching from one firm to another without any priced transaction. This "non-priced" diffusion process particularly characterizes military R&D activities, which generate almost immediate effects in the military sector and next in the civilian sector (Benoit [1973], [1978])

The first attempts to achieve an economic modeling of the dynamic economic effects of military R&D, follow the developments, at the end of the 80's, of the so-called *New Growth Theories*, which consider the economic growth rate as an endogenous variable depending on the fundamentals of the economy. As sketched above, at a very rough level, two main effects of military research on economic activity can be highlighted:

(i) On the one hand, military R&D and its applications improve the performance of military equipment *i.e.* the quality of national defense services provided by the army and, eventually, the global welfare.

(ii) On the other hand, military R&D increases the total stock of available scientific and technical knowledge, diffusing sooner or later, to the overall economy and, eventually, contributing to a more efficient production process which results in a higher growth rate.

The first mechanism directly affects the consumers' utility function and the second the aggregate production function: military R&D spending, through innovations diffusion, generates spillovers effects from the military sector towards the whole productive system. These positive externalities constitute a fundamental non-priced productive factor generating increasing returns and endogenous growth. Because a particular firm does not take into account, when making its R&D investments decisions, the positive impact of such investments on other firms and the overall economy, total R&D spending stands far below its socially optimal level; the role of the Government is thus to design the appropriate incentive schemes to encourage firms of the military sector to sufficiently invest into R&D.

The introduction of National defense services as a component of the utility function is an old idea – Brito, [1972], Deger & Sen [1984], Van der Ploeg & Zeeuw, [1990], Zou, [1995], Chang & *alii* [1996] – providing an economic modeling of the *Demand Side* impact of military expenditures. The long-run *Supply Side* effects of military R&D have not been widely analyzed in economic literature even if one can find few theoretical papers devoted to formalize the effects of military R&D operating as an external production factor. In line with Barro [1990] seminal paper – considering public spending as an external growth factor and evaluating its impact on GDP growth – Shieh & *alii* [2002] develop a dynamic general equilibrium model where the military spending – research and substructures – is analyzed as an external effect "doping" the production function and generating a self-maintained growth. The model developed in this paper, in line with Shieh & *alii* [2002], differs from the latter in four main points:

(i) To compute the optimal share of military R&D investments into public spending and military spending, we distinguish four main public spending components: civilian consumption, civilian investment, ordinary milex and military R&D.[3]

(ii) To investigate the double nature of military R&D, we carefully distinguish the direct effect of military R&D on the utility function – through the quality of defense services – from its supply side impact on the aggregate production function

(iii) Depreciation of capital and public spending is allowed.

(iv) A numerical simulation of the model on French data allows us to give an evaluation of the impact of an increase in military R&D investments on GDP and to compute the optimal relative size of military R&D.

After the presentation of the model (section 2), and the equilibrium (section 3), section 4 is devoted to the dynamic analysis. Eventually, section 5 deals with economic policy analysis.

[3] Shieh & *alii* [2002] only deal with civilian *vs* military public spending.

2 The model

The main purpose of the model developed in this section is to provide a simple economic framework to address an economic policy question, namely the optimal relative size of military R&D investment within total public expenditures. In order to be able to provide such an estimation, one needs first to understand how military R&D investments affect the global welfare; two main effects can be distinguished. On one hand, military R&D and its applications raise the army productivity *i.e.* the quality of national defense services provided by the army: this first R&D "transmission channel" from the defense sector to the civil one directly affects the welfare through the utility function of economic agents. On the other hand, military R&D investments increase the total stock of scientific and technical knowledge available in the economy and, consequently, positively affect inputs productivity (externalities) and eventually the growth rate of the economy and the welfare: this indirect effect constitute a second R&D "transmission channel" from the defense sector to the civil one.

To capture the long run impact of military R&D on the growth rate of the economy, we develop an endogenous growth model in line with Barro [1990] and Shieh & *alii* [2002]. The following subsections present the comportments of the three agents of the model.

2.1 Households

Households are supposed to live an infinite number of periods during which they consume a private consumption good c, a public consumption good b and a national defense public service denoted by e. The overall level of utility reached by the representative household during his life is given by the intertemporal utility function:

$$\sum_{t=0}^{\infty} \beta^t \left[u\left(c_t\right) + v\left(b_t\right) + w\left(e_t\right) \right] \tag{1}$$

where $0 < \beta \equiv 1/\left(1+i\right) < 1$ denotes the discount factor and $i > 0$ the time preference rate.

We split the total military expenditures into two components – standard military expenditures n and R&D military expenditures m – and assume that each of them is used to produce the national defense public good $e_t \equiv e\left(m_t, n_t\right)$; the function $e\left(.\right)$ is supposed to have constant returns to scale: $e\left(\mu m, \mu n\right) = \mu e\left(m, n\right)$.

Such a distinction allows us to further analyze the specific role played by military R&D investments and their effects on the global economy and the welfare.

At each period of time the representative household faces the following budget constraint:

$$c_t + \left(k_{t+1} - \Delta_k k_t\right) \leq \left(1-\tau\right)\left(r_t k_t + \omega_t l_t\right) \tag{2}$$

where $\Delta_k \equiv 1 - \delta_k$ denotes the capital depreciation rate between two periods.

Consumption and investment net expenditures stand on the left side of equation (2) while on the right side figures the disposable income with r the real return on capital, ω the real wage rate and τ the tax rate.

Labor supply is assumed to be inelastic and normalized to one:[4]

$$l_t = 1 \tag{3}$$

In such a framework the consumer problem is to maximize the intertemporal utility function (1) with respect to k_t, and c_t. The infinite horizon Lagrangian function can thus be written:

$$\sum_{t=0}^{\infty} \beta^t \left[u\left(c_t\right) + v\left(b_t\right) + w\left(e_t\right)\right] + \sum_{t=0}^{\infty} \lambda_t \left[\left(1-\tau\right)\left(r_t k_t + \omega_t\right) - c_t - k_{t+1} + \Delta_k k_t\right]$$

After elimination of Lagrange multipliers, first order conditions lead directly to the Euler equation,

$$\frac{u'\left(c_t\right)}{u'\left(c_{t+1}\right)} = \beta\left[\Delta_k + \left(1-\tau\right) r_{t+1}\right] \tag{4}$$

and the budget constraint (2), rewritten as an equality.

Moreover, the optimal solution must respect the following transversality condition:

$$\lim_{t\to\infty} \lambda_t k_{t+1} = 0 \tag{5}$$

Assumption 1. *The utility function is characterized by a constant ε_u elasticity of intertemporal substitution in consumption:*

$$u\left(c\right) \equiv c_u \frac{c^{1-1/\varepsilon_u} - 1}{1 - 1/\varepsilon_u} \tag{6}$$

where c_u is an arbitrary constant.

2.2 Firms

The state of technology is represented by a production function including four inputs: the capital k, the labor l, a public investment good and military R&D expenditures m. The public investment good is summarized by the amount a of public expenditures devoted to raise the quantity and/or quality of education and public substructures as roads, airports, cable networks etc. (productive externalities). On their side, military R&D expenditures affect the global productivity through a standard R&D externality (spillovers effects from the defense sector to the civil sector).

Assumption 2. *The production function $F\left(k, l, a, m\right)$ exhibits labor and capital constant returns to scale:*

$$F\left(\mu k, \mu l, a, m\right) = \mu F\left(k, l, a, m\right)$$

[4] A more general approach would, of course, introduce the labor disutility into the utility function, but such a change would not affect the main results of the model.

The intensive production function $f(\kappa, a, m) \equiv F(k/l, 1, a, m)$ *is assumed to be homogeneous of degree one with respect to its variables:* $f(\mu\kappa, \mu a, \mu m) = \mu f(\kappa, a, m)$.

The producer problem is to maximize the profit with respect to capital stock k_t and labor force l_t, considering all public goods externalities – *i.e.* a and m - as constants

$$\max_{k_t, l_t} F(k_t, l_t, a_t, m_t) - r_t k_t - \omega_t l_t$$

The firm equilibrium is thus defined by the equality, for each input, between its real cost and its productivity:

$$\begin{aligned} r &= F_k(k_t, l_t, a_t, m_t) \\ \omega &= F_l(k_t, l_t, a_t, m_t) \end{aligned}$$

These equalities can be rewritten using the intensive production function:

$$\begin{aligned} r_t &= f_\kappa(\kappa_t, a_t, m_t) \\ \omega_t &= f(\kappa_t, a_t, m_t) - \kappa f_\kappa(\kappa_t, a_t, m_t) \end{aligned} \tag{7}$$

2.3 Government

As already mentioned, one assumes that the total amount of public expenditures is constituted of civil investment a (public networks substructures, education), public consumption b (health, justice, employment and social policies, etc.), military R&D investment m and standard military spending n (arms, troops, buildings etc.):

$$g_t \equiv a_t + b_t + m_t + n_t$$

The Government budget constraint at time t is thus given by:

$$a_{t+1} - \Delta_a a_t + b_{t+1} - \Delta_b b_t + m_{t+1} - \Delta_m m_t + n_{t+1} - \Delta_n n_t \leq \tau(r_t k_t + \omega_t l_t) \tag{8}$$

where $\Delta_i \equiv 1 - \delta_i$ with δ_i the depreciation rate of public expenditure of type i; the right side of (8) represents the total amount of taxes.[5]

In such a model the economic policy is simply described by the overall tax rate τ and the breakdown of fiscal revenues into the four components of public spending:

$$(\sigma_a, \sigma_b, \sigma_m, \sigma_n) \equiv (a_t/g_t, b_t/g_t, m_t/g_t, n_t/g_t) \tag{9}$$

where

$$\sigma_a + \sigma_b + \sigma_m + \sigma_n = 1 \tag{10}$$

Using the key (9), and the budget constraint, equation (8) can be rewritten:

$$\begin{aligned} & \sigma_a[g_{t+1} - \Delta_a g_t] + \sigma_b[g_{t+1} - \Delta_b g_t] + \sigma_m[g_{t+1} - \Delta_m g_t] + \sigma_n[g_{t+1} - \Delta_n g_t] \\ = \; & g_{t+1} - [\sigma_a\Delta_a + \sigma_b\Delta_b + \sigma_m\Delta_m + \sigma_n\Delta_n] g_t \\ \leq \; & \tau(r_t k_t + \omega_t l_t) \end{aligned}$$

[5] A lag can be introduced between fiscal revenues and public expenditures, but this does not change the long term analysis and the stationary state of the model.

or equivalently:

$$g_{t+1} - \Delta g_t \leq \tau \left(r_t k_t + \omega_t l_t \right) \tag{11}$$

Where the depreciation factor of global public expenditure is now the weighted average of all previous specific depreciation factors:

$$\Delta \equiv \sigma_a \Delta_a + \sigma_b \Delta_b + \sigma_m \Delta_m + \sigma_n \Delta_n \tag{12}$$

3 Equilibrium

Equilibrium of the labor market being characterized by an inelastic labor supply (*cf.* (3)), the general equilibrium of the model requires equilibrium on both goods and inputs markets. Noticing that $r_t k_t + \omega_t l_t = r_t \kappa_t + \omega_t = f(\kappa, a, m)$ and that $a_t = \sigma_a g_t$ and $m_t = \sigma_m g_t$, one easily rewrites the representative agent budget constraint (2) as an aggregate resources constraint:

$$c_t + \kappa_{t+1} - \Delta_k \kappa_t = (1 - \tau) f(\kappa_t, \sigma_a g_t, \sigma_m g_t) \tag{13}$$

while the Government budget constraint (11) becomes:

$$g_{t+1} - \Delta g_t = \tau f(\kappa_t, \sigma_a g_t, \sigma_m g_t) \tag{14}$$

Substituting (7) in Euler equation (4), one gets:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta \left[\Delta_k + (1 - \tau) f_\kappa (\kappa_{t+1}, \sigma_a g_{t+1}, \sigma_m g_{t+1}) \right] \tag{15}$$

Noticing that the homogeneity property of the intensive production function implies that its derivatives are homogeneous of degree zero,

$$f_\kappa(\mu\kappa, \mu a, \mu m) = f_\kappa(\kappa, a, m)$$

it comes immediately $f_\kappa(\kappa_t, \sigma_a g_t, \sigma_m g_t) = f_\kappa(\kappa_t/g_t, \sigma_a, \sigma_m)$. Defining thus,

$$\begin{aligned} x_t &\equiv \kappa_t / g_t \\ \varphi(x_t) &\equiv f(\kappa_t/g_t, \sigma_a, \sigma_m) \end{aligned}$$

it is easy to show that, $\varphi'(x_t) = f_\kappa(\kappa_t, \sigma_a g_t, \sigma_m g_t)$ which implies: $r_t = \varphi'(x_t) = f_\kappa(x_t, \sigma_a, \sigma_m)$. Equations (13), (14) and (15) can thus be rewritten:

$$\begin{aligned} c_t + \kappa_{t+1} - \Delta_k \kappa_t &= (1 - \tau) g_t \varphi(x_t) && (16) \\ g_{t+1} - \Delta g_t &= \tau g_t \varphi(x_t) && (17) \\ \frac{u'(c_t)}{u'(c_{t+1})} &= \beta \left[\Delta_k + (1 - \tau) \varphi'(x_{t+1}) \right] \end{aligned}$$

Defining,

$$\begin{aligned} y_t &\equiv c_t / g_t && (18) \\ \gamma_t &\equiv g_{t+1} / g_t \end{aligned}$$

and dividing both sides of (16) and (17) by g_t, one eventually gets:

$$y_t + \gamma_t x_{t+1} - \Delta_k x_t = (1-\tau)\varphi(x_t) \quad (19)$$
$$\gamma_t = \Delta + \tau\varphi(x_t) \quad (20)$$

On the other hand, the Euler equation can be rewritten under Assumption 1:

$$c_{t+1}/c_t = (\beta[\Delta_k + (1-\tau)\varphi'(x_{t+1})])^{\varepsilon_u}$$

We thus have:

$$\frac{y_{t+1}}{y_t}\gamma_t = (\beta[\Delta_k + (1-\tau)\varphi'(x_{t+1})])^{\varepsilon_u} \quad (21)$$

4 The dynamic system

Substituting (20) into (21) and (19), one easily gets:

$$[\Delta + \tau\varphi(x_t)]\frac{y_{t+1}}{y_t} = (\beta[\Delta_k + (1-\tau)\varphi'(x_{t+1})])^{\varepsilon_u} \quad (22)$$
$$y_t - \Delta_k x_t + \Delta x_{t+1} = (1-\tau-\tau x_{t+1})\varphi(x_t) \quad (23)$$

These equations constitute a two-dimensional dynamic system in (x_t, y_t) where x_t – but not y_t – is a predetermined variable.

4.1 The stationary state

The steady-state evaluation of the above dynamic system gives:

$$\gamma = \Delta + \tau\varphi(x) = (\beta[\Delta_k + (1-\tau)\varphi'(x)])^{\varepsilon_u} \quad (24)$$
$$y = (\Delta_k - \Delta)x + (1-\tau-\tau x)\varphi(x) \quad (25)$$

It is easy to show that one necessarily have a regular growth at the stationary state: $\gamma \equiv g_{t+1}/g_t = c_{t+1}/c_t = k_{t+1}/k_t$. Noticing that $\lambda_t = \beta^t u'(c_t)$ and using (6), the transversality condition (5) becomes:

$$\lim_{t\to\infty} c_u c_0^{-1/\varepsilon_u} k_0 \gamma \left(\beta\gamma^{1-1/\varepsilon_u}\right)^t = 0$$

i.e. $\beta\gamma^{1-1/\varepsilon_u} < 1$. We thus get $\gamma < \Delta_k + \rho$ from (24), where $\rho \equiv (1-\tau)r$ is the capital net return after taxes.

4.2 Local dynamics

The question of the uniqueness of general equilibrium under rational expectations is a main theoretical issue. In this section one shows that the equilibrium is unique and constitutes a saddle path converging to the stationary state. The starting point for the path is defined by the predetermined variable x_0, while the adjustment of the non-predetermined variable y_0 guarantees that the starting point lies to the converging saddle path, which is the only path compatible with the long-run equilibrium (satisfaction of the transversality condition and non-negative variables). To show the saddle point stability, we will now analyze the local dynamics by linearizing the dynamic system around the steady state.

Differentiating equation (22) w.r.t. the dynamic variables $(x_{t+1}, y_{t+1}, x_t, y_t)$, using (24-25), one gets,

$$\gamma\varepsilon_u \frac{\varphi'' x}{\varphi'} \frac{(1-\tau)\varphi'}{\Delta_k + (1-\tau)\varphi'} \frac{dx_{t+1}}{x} - \gamma\frac{dy_{t+1}}{y} = \tau\varphi' x \frac{dx_t}{x} - \gamma\frac{dy_t}{y} \tag{26}$$

where all partial derivatives have been normalized using the stationary state.

Linearizing now equation (23) around the steady state, one gets:

$$\gamma\frac{dx_{t+1}}{x} = [\Delta_k + (1-\tau-\tau x)\varphi'] \frac{dx_t}{x} - \frac{y}{x}\frac{dy_t}{y} \tag{27}$$

Denoting $\varepsilon_2 \equiv x\varphi''/\varphi' < 0$ the interest rate elasticity with respect to the ratio κ/g (capital per capita over public spending), the linear system (26-27) can be rewritten using the Jacobian matrix:

$$\begin{bmatrix} \frac{dx_{t+1}}{x} \\ \frac{dy_{t+1}}{y} \end{bmatrix} = \begin{bmatrix} \gamma\varepsilon_u\varepsilon_2\frac{\rho}{\Delta_k+\rho} & -\gamma \\ \gamma & 0 \end{bmatrix}^{-1} \begin{bmatrix} x\rho\frac{\tau}{1-\tau} & -\gamma \\ \Delta_k + \rho\frac{1-(1+x)\tau}{1-\tau} & -\frac{y}{x} \end{bmatrix} \begin{bmatrix} \frac{dx_t}{x} \\ \frac{dy_t}{y} \end{bmatrix}$$

The determinant and the trace of the Jacobian matrix are respectively:

$$D = \frac{\Delta_k + \rho}{\gamma} - \frac{\rho}{\gamma}\frac{\tau}{1-\tau}\left(\frac{y}{\gamma} + x\right) \tag{28}$$

$$T = 1 + D + \frac{\rho}{\gamma}\left(\frac{\tau}{1-\tau}\frac{y}{\gamma} - \frac{\varepsilon_u\varepsilon_2}{\Delta_k+\rho}\frac{y}{x}\right) \tag{29}$$

Proposition 1 establishes the uniqueness of the equilibrium transition.

Proposition 1 *The equilibrium is unique.*

Proof. *Cf.* Section 7 - Appendix.

Proposition 2 below, allow us to explicitly characterize the stability under a simplifying hypothesis: to demonstrate the saddle path stability, we now assume that the weighted average of the public expenditures depreciation rates is simply equal to the capital depreciation rate.

Assumption 3. $\Delta = \Delta_k$.

Proposition 2 *The stationary state is a saddle point.*

Proposition 2 confirms Barro [1990] intuition. In this seminal paper one gets a dynamic system of one equation including one non-predetermined variable; the model exhibits one instable – determined – stationary state constituting the only possible equilibrium. In our model, as in Barro [1990], the equilibrium is determined, but an equilibrium transition is possible too.

5 Economic policy

Shieh & *alii* [2002] analyze how the growth rate of the economy and the global welfare depend on the relative weights of civilian *vs* military expenditures into public spending. They show that it does exist an optimal ratio, Military expenditures/GDP, which maximizes the economic growth but highlight that this ratio stands below the ratio maximizing the global welfare. This result contributes to explain that in some countries, military expenditures cuts associated with disarmament policies can reduce global welfare.

In our model, one focuses on the optimal sharing – welfare and growth maximizing – of public resources between civilian investments a, public consumption b, military R&D investment m and "standard" military spending n. In the previous sections economic agents solve their programs considering as given the Government economic policy $(\tau, \sigma_a, \sigma_b, \sigma_m, \sigma_n)$ *i.e.* the tax rate and the breakdown of fiscal revenues into the four components defined above; in such a situation the main problem for the Government is to determine the optimal economic policy *i.e.* $(\tau^*, \sigma_a^*, \sigma_b^*, \sigma_m^*, \sigma_n^*)$. One can notice that in a representative agent framework, maximizing any social welfare function (growing function of personal utilities) is equivalent to maximize the utility of the representative agent – function (1) – with respect to the five economic policy tools.

To simplify the problem, one focuses only on the case of an economy characterized by a regular economic growth *i.e.* at a long run equilibrium close to the stationary state.[6] It is clear here, as in Shieh & *alii* [2002], that maximizing the economic growth is not equivalent to social welfare maximizing.

As already seen, one assumes a Cobb-Douglas production function homogeneous of degree one (Assumption 2 above) in order to easily get numerical simulations from the model:

$$\begin{aligned} F(\mu k, \mu l, a, m) &= \mu F(k, l, a, m) \\ f(\mu \kappa, \mu a, \mu m) &= \mu f(\kappa, a, m) \end{aligned}$$

For similar reasons on assumes a Cobb-Douglas production function for the defense good.

Assumption 4. *The general production function $F(.)$ and the defense good production function $e(.)$ can be written:*

$$\begin{aligned} F(k, l, a, m) &= \theta k^{\alpha} l^{1-\alpha} a^{\alpha_a} m^{\alpha_m} \\ e(\sigma_m, \sigma_n) &\equiv B \sigma_m^{\beta_m} \sigma_n^{\beta_n} \end{aligned}$$

with $\alpha + \alpha_a + \alpha_m = 1$ *and* $\beta_m + \beta_n = 1$.

[6] In fact, because of the uniqueness of the equilibrium, we could compute utility on a transitional path, when the starting point stands off the steady state, and maximize its value with respect to economic policy parameters, but it would drive us to hard analytical computations. As the main goal of this paper is to analyze long-run economic policy effects, one can focus on the stationary state and transitional equilibria close enough to the steady state (under a continuity hypothesis, optimal economic policy rules does not change if the equilibrium path stands in an area close enough to the stationary state).

Moreover, we substitute Assumption 5 below to previous Assumption 3; this new and stronger assumption implies $\Delta = \Delta_k$ (Assumption 3), but sets new constraints on public expenditure depreciation factors.

Assumption 5. $\Delta_a = \Delta_b = \Delta_m = \Delta_n = \Delta_k$.

We eventually restrict ourselves to the case of a logarithmic utility function, more easy to handle and widely accepted both by economists and RBC econometricians:

Assumption 6. $u(c) \equiv c_u \ln c$, $v(b) \equiv c_v \ln b$, $w(e) \equiv c_w \ln e$.

Using a logarithmic utility function implies setting the intertemporal elasticity of substitution equal to one; the social welfare function can then be written:

$$W = \sum_{t=0}^{\infty} \beta^t c_u \ln c_t + \sum_{t=0}^{\infty} \beta^t c_v \ln b_t + \sum_{t=0}^{\infty} \beta^t c_w \ln e_t$$

where, without loss of generality:

$$c_u + c_v + c_w = 1 \tag{30}$$

Proposition 3 *Under Assumptions 4, 5 and 6, the optimal economic policy is obtained by solving for x the following implicit equation:*

$$\theta \sigma_a^{\alpha_a} \sigma_m^{\alpha_m} x^{\alpha} = \frac{x\Delta (1-\beta)}{\beta \varepsilon_1 (1-\tau) - \tau x} \tag{31}$$

where τ is defined by

$$\begin{aligned} \tau(x) &= \left[1 + A - B \pm \sqrt{(1+A-B)^2 - 4A} \right] /2 \qquad (32) \\ A &\equiv \frac{\varepsilon_1 (\varepsilon_1 c_u (\beta + x) + (1-\varepsilon_1)(\varepsilon_1 c_u - 1)\beta x)}{c_u (\varepsilon_1 + x)(\beta\varepsilon_1 + 2\beta\varepsilon_1 x + x^2) - \beta\varepsilon_1 x (1+x)} \\ B &\equiv \frac{x^2 ((c_u (1+\varepsilon_1) - \varepsilon_1)\beta\varepsilon_1 + c_u (x - \varepsilon_1))}{c_u (\varepsilon_1 + x)(\beta\varepsilon_1 + 2\beta\varepsilon_1 x + x^2) - \beta\varepsilon_1 x (1+x)} \end{aligned}$$

and σ_a, σ_m, σ_n are given by:

$$\sigma_a = \left(1 + \frac{\alpha_m}{\alpha_a} + \frac{\varepsilon_1}{\alpha_a} \frac{[c_v + (\beta_n + \beta_m) c_w][\beta(1-\varepsilon_1)(1-\tau) + \tau x]}{c_u + c_u \frac{\tau x}{\tau x - (1-\tau)} + \frac{\beta}{1-\beta}\frac{\tau x}{\tau x - (1-\tau)\varepsilon_1} - \left(c_u + \frac{\beta}{1-\beta}\right)\frac{\tau x}{\tau x - (1-\tau)\beta\varepsilon_1}} \right)^{-1} \tag{33}$$

$$\sigma_m = \frac{\alpha_m (c_v + \beta_n c_w) - \alpha_a \beta_m c_w}{\alpha_a (c_v + \beta_n c_w + \beta_m c_w)} \sigma_a + \frac{\alpha_a \beta_m c_w}{\alpha_a (c_v + \beta_n c_w + \beta_m c_w)} \tag{34}$$

$$\sigma_n = -\frac{(\alpha_a + \alpha_m)\beta_n c_w}{\alpha_a (c_v + \beta_n c_w + \beta_m c_w)} \sigma_a + \frac{\beta_n c_w}{c_v + \beta_n c_w + \beta_m c_w} \tag{35}$$

After getting x^*, one computes τ^* from (32), σ_a^* from (33) and, eventually, σ_m^*, σ_n^* and σ_b^* from (34), (35) and (10).

Proof. *Cf.* Section 7 – Appendix.

Let us notice that (31) is an implicit equation easy to solve by numerical methods, considering as constants the fundamental parameters of the economy.

6 Numerical simulations

The purpose of this subsection is to justify the parameters values we used for simulating the model and to present the main results concerning the impact on GDP of increasing military R&D public investment.

6.1 Parametrization and calibration

We assume a 7.76% annual depreciation rate – corresponding, more or less, to a 50% depreciation in 9 years[7] – and a rate of time preference equal to 6.2%.[8] The share α of capital remuneration in GDP is set to 75% according to the results of Mankiw & *alii* [1992] and most of the empirical estimations; α is a mesure of both human and physical capital share in total income, while $1 - \alpha$ is the weight of productive externalities.

One goal of the paper is to analyze the impact of military R&D on economic growth and global welfare; as noticed in the previous section, such an analysis highlights the crucial role played by productive externalities associated to R&D public investment. In order to get a *prudential evaluation* of the economic impact of an increase in public military R&D expenditures and to avoid any overestimation of this impact, we minimize the size of military R&D externalities by setting $\alpha_m = 0.5\%$.

For a similar reason we decided:

(i) to limit the relative weight of defense services in the household utility function – *i.e.* the impact of military R&D on global welfare – by considering that households strongly prefer non-military goods to the military one : $c_u = c_v = 45\%$, $c_w = 10\%$.

(ii) to limit the role played by military R&D in the Defense good production function: $\beta_m = 25\%$.

This set of very cautious and pessimistic assumptions, concerning the role played by military R&D in the global economy, should protect us against any overestimation of its impact on GDP and the results obtained can probably be considered as lower bounds.

The annual real growth rate of the economy has been set to 1.5% and the global fiscal pressure is supposed to be equal to 40% ; the relative weights $\sigma_a, \sigma_b, \sigma_m, \sigma_n$ of public spending components simply correspond to the observed values on the French economy:

$$(\tau, \sigma_a, \sigma_b, \sigma_m, \sigma_n) = (0.4, 0.179, 0.761, 0.002, 0.058) \tag{36}$$

Considering that

$$\gamma = \Delta + \tau\theta\sigma_a^{\alpha_a}\sigma_m^{\alpha_m}x^{\alpha} \tag{37}$$

[7] According to empirical avalaible estimations, yearly capital depreciation rate stands around 1.5% for human capital and 9% for physical capital. Aghion & Howitt [1997] consider a composite human and physical capital with a unique depreciation rate of 8%.

[8] Corresponding to an annual discount factor $\beta = 0.94$.

and using (31), one gets:

$$\theta = \frac{\gamma - \Delta}{\tau \sigma_a^{\alpha_a} \sigma_m^{\alpha_m}} \left[\varepsilon_1 \frac{\beta(\gamma - \Delta)}{\gamma - \beta\Delta} \frac{1-\tau}{\tau} \right]^{-\alpha}$$

The previous set of assumptions concerning the growth rate of the economy, the fiscal pressure and the relative weights of public spending components, eventually leads to set the value of the global productivity parameter: $\theta = 0.124$.

6.2 Increasing military R&D under constant global public spending

In this section one proceeds to numerical simulations of the model to address the following question: what is the impact on the growth rate of the economy and the GDP, of doubling the military R&D share into public spending σ_m,[9] while decreasing at the same time the public consumption share σ_b in order to keep constant global public expenditures? In other words: would it be optimal, from an economic policy point of view, to switch some fiscal resources from civilian unproductive spending to military R&D investments. One must notice that such a switch represents a decrease of σ_b from 0.761 to 0.759 *i.e.* a -0.26% slight variation in the relative weight of public consumption.

Formally the problem is to provide an evaluation of the GDP increase involved by a $d\sigma_m$ increase of σ_m coming with a joint decrease $d\sigma_b = -d\sigma_m$ of public unproductive consumption. Setting $d\sigma_m = -d\sigma_b$ and $d\tau = d\sigma_a = d\sigma_b = 0$, differentiating (37) and noticing that $\varphi(x, \sigma_a, \sigma_m) = \theta \sigma_a^{\alpha_a} \sigma_m^{\alpha_m} x^{\alpha}$, one gets in the long run (after a transitional adjustment):

$$d\gamma = \tau \varphi_m d\sigma_m + \tau \varphi_x dx \tag{38}$$

Differentiating totally,

$$\varphi(x, \sigma_a, \sigma_m) = \frac{(1-\beta)\Delta x}{\beta \varepsilon_1 (1-\tau) - \tau x} \tag{39}$$

we have $dx = x_\tau d\tau + x_a d\sigma_a + x_m d\sigma_m$ where $d\tau = d\sigma_a = 0$; using equation (58) given in Appendix, dx can then be written:

$$dx = \frac{\alpha_m}{\sigma_m} \frac{x}{\varepsilon_1} \frac{1}{\beta(1-\varepsilon_1)(1-\tau) + \tau x} d\sigma_m \tag{40}$$

Substituting (40) into (38) one gets the impact of the military R&D increase on the quarterly growth factor of the economy:

$$\frac{d\gamma}{d\sigma_m} = (\gamma - \Delta) \frac{\alpha_m}{\sigma_m} \left[1 + \frac{1}{\beta(1-\varepsilon_1)(1-\tau) + \tau x} \right] > 0$$

[9] *i.e.* a shift from 0.002 to 0.004 of σ_m.

Using the parameterization presented in section 6.1.1 and noticing that the following equation stands at the stationary state:

$$x = \left(\frac{\gamma - \Delta}{\tau \theta \sigma_a^{\alpha_a} \sigma_m^{\alpha_m}} \right)^{1/\alpha}$$

We eventually get:

$$\gamma' \approx \gamma + \frac{d\gamma}{d\sigma_m} \left(\sigma'_m - \sigma_m \right) = 1.0166$$

where γ' represents the yearly stationary growth factor, corresponding to the new share of military R&D into global public spending : $\sigma'_m = 0.004$. To summarize: if the government would decide to double the share of military R&D into public spending, while keeping the latter constant, through a decrease of the unproductive public consumption share, the growth rate of the economy would shift from 1.5% to more than 1.66%[10] corresponding to a 2.7 billions euros GDP benefit for one full year (for a slight $-0,26\%$ decrease in public consumption).

Noticing that a permanent increase in the relative weight of military R&D, affects not only the current but the permanent growth rate of the economy, one needs to compute the discounted value of all future GDP increases associated with the initial change.

Let us note z_0 the GDP of the current year, z_t the GDP of year t and z'_t the GDP after the increase in the share of military R&D: $\sigma'_m = 2\sigma_m$; the GDP benefit associated to the economic policy change is equal, at time t, to: $z'_t - z_t$. Denoting γ' the annual growth rate corresponding to σ'_m, we have $\gamma' = \beta R'$ (Euler equation) along the new regular growth path, where $R' \equiv \Delta + (1 - \tau) r'$; at the stationary state, we just need to discount future GDP benefits, to get the intertemporal benefit Γ:

$$\Gamma \equiv \sum_{t=0}^{\infty} \frac{z'_t - z_t}{R'^t}$$

By comparison of the regular growth rates $z_t = z_0 \gamma^t$ and $z'_t = z_0 \gamma'^t$, one easily gets:

$$\Gamma = z_0 \frac{\beta}{1 - \beta} \frac{\gamma' - \gamma}{\gamma' - \beta\gamma} \approx 684$$

Considering that $z_0 = 1648$ billions euros (French 2004 GDP), one eventually gets an overall GDP benefit – the sum of all discounted annual benefits – standing around 684 billions euros. The discounted benefit for the only year 2005 is equal to 2.64 billions euros:

$$\Gamma_{2005} = z'_1 - z_1 = (\gamma' - \gamma) z_0 \approx 2.64$$

A limited 1.32 billions euros permanent reallocation of public spending, $(\sigma'_m - \sigma_m) \tau z_0 = 1.32$, from civilian unproductive public consumption toward military R&D investment, induces a 2.64 billions euros discounted benefit the first year and a

[10] *i.e.* 11% raise of the growth rate.

94 billions euros discounted benefit on a decade:

$$\Gamma \equiv \sum_{t=0}^{10} \frac{z'_t - z_t}{R'^t} = z_0 \sum_{t=0}^{10} \frac{\gamma'^t - \gamma^t}{(\gamma'/\beta)^t} \approx 94$$

It is worth noting that this result would remain the same with a budget reallocation going from military non-R&D expenditures toward military R&D; from this point of view a decrease in σ_n is strictly equivalent to a decrease in σ_b as far as it is used to double the relative weight σ_m of military R&D inside global spending. However the decrease in σ_n, from 0.058 to 0.056 (-3.45%), which is necessary to remain constant the global public spending despite the increase of military R&D investment, is higher than the corresponding 0.761 to 0.759 decrease of σ_b (-0.26%). In other words, it is always possible, through an internal reallocation of the defense budget, to reach the same level of benefits than one could get with a budget reallocation going from civilian expenditures toward military R&D, but this implies a higher level of "effort" for the military sector than for the civilian sector.

6.3 Optimal policy

The purpose of this subsection is to compute the optimal economic policy *i.e.* the tax rate and the public spending shares – $(\tau^*, \sigma_a^*, \sigma_b^*, \sigma_m^*, \sigma_n^*)$ – maximizing the stationary state global welfare, considering as given all the exogenous parameters: $\beta = 0.941$, $\delta = 0.0776$, $\alpha_a = 0.245$, $\alpha_m = 0.005$, $c_u = c_v = 0.45$, $c_w = 0.1$, $\beta_m = 0,25$ and $\theta = 0.124$. Applying the method described in Proposition 3, one gets $x^* = 3.765$ and eventually:

$$(\tau^*, \sigma_a^*, \sigma_b^*, \sigma_m^*, \sigma_n^*) = (0.126, 0.887\,, 0.078\,, 0.022, 0.013\,)$$

A tax rate under 13%, leads to an optimal growth factor γ equal to 1.083, corresponding to an annual growth rate around 8%.

Despite a "prudential" set of assumptions (*cf.* subsection 6.1.1) concerning the productive impacts of military R&D, a small effect of the defense good into the utility function and equal weights for private and public consumption goods ($c_u = c_v$), one gets a large gap between "productive" expenditures on one hand ($\sigma_a = 88.7\%$, $\sigma_m = 2.2\%$) and non-productive public expenditures on the other hand ($\sigma_b = 7.8\%$, $\sigma_n = 1.3\%$). This result is a well known characteristic of endogenous growth models where productive externalities play a central role: in such models, investments in R&D, education or public substructures are associated with high long-run returns because they induce productive externalities which durably improve the efficiency of the production process and, consequently, the social welfare. In such an economic word, if policy makers do not care with electoral cycles, but only with long-run growth and welfare, they must minimize the level of unproductive investments,[11] to strongly invest in R&D, networks substructures and education.

[11] As, for example, civilian or military bureaucracies.

7 Conclusion

The general equilibrium endogenous growth model presented in this paper emphasizes the key role played by public military R&D investments in determining the long run levels of economic growth and welfare. While inspired by Shieh & *alii* [2002], it departs from the latter by using more general specifications (CES functions), distinguishing unproductive defense services from military R&D, taking into account the impact – through the quality of national defense – of military research on the household utility function, allowing us to compute the optimal degree of military R&D and to proceed to a numerical simulation on French data.

From a theoretical point of view we obtain four main results: *(i)* the equilibrium path is unique, *(ii)* market imperfections (externalities + taxes) make the equilibrium inefficient under an arbitrary policy, *(iii)* an opportune fiscal policy (tax rate + public spending shares) can restore the second best, *(iv)* military research matters more than the ordinary milex in order to achieve the social welfare target in the long run. This last point is crucial as it stresses how the military research, as a productive externality, is a powerful engine for growth, compared to alternative policies, such as, public consumption or ordinary milex. This depends on the fact that in an endogenous growth framework, knowledge accumulation has a dramatic and unbounded impact on factors productivity.

A numerical simulation, based on a pessimistic set of assumptions concerning the global impact of military R&D, shows that a slight 1.32 billions euros permanent reallocation of public spending from civilian unproductive public consumption toward military R&D investment, induces a 2.6 billions euros GDP benefit for the first year and a discounted benefit on a decade close to 100 billions euros. In such a framework, characterized by productive externalities originating in military R&D, the Government optimal economic policy should be to massively invest in military R&D.

Even if the previous numerical simulation can be seen as a fruitful illustration of what could be the benefits associated to a moderate public spending reallocation in favor of military R&D, a more general and reliable model should include a full description of the strategic interactions between countries and internalize the negative long run externalities potentially associated with high-tech arms production.

On one hand one needs to analyze how technology and industrial leaks, through innovations diffusion, can generate international impacts of local military R&D. For example, military R&D investment and endogenous growth could be combined in a dynamic model of capital and arms accumulation, *à la* Zou [1995], using dynamic game theory to modelize arms racing and strategic interactions between countries.

On the other hand military R&D contributes to the development of high-tech arms – biological or nanotechnological weapons, robotic arms etc. – which can eventually be export toward under-developed countries characterized by imperfect arms controls; in a new mass terrorism environment this long-run negative externality, must certainly be introduced in the previous analysis.

8 Appendix

Proof of Proposition 1. We need to prove that the equilibrium is always determined; for that the dimension of the stable variety must be smaller than the number of predetermined variables which is equal to one. In other words one must rule out any sink configuration (characterized by two eigenvalues inside the unit circle). As $\varepsilon_2 < 0$, (29) implies:

$$D = T - 1 - \frac{\rho y}{\gamma}\left(\frac{\tau}{1-\tau}\frac{1}{\gamma} - \frac{1}{x}\frac{\varepsilon_u \varepsilon_2}{\Delta_k + \rho}\right) < T - 1 \tag{41}$$

It is easy to show that the two eigenvalues lie inside the unit circle if and only if $D < 1$, $D > T - 1$ and $D < -T - 1$. As, inequality (41) violates the second condition any sink configuration is impossible: at least one eigenvalue lies outside the unit circle and the uniqueness of the equilibrium, if any, is guaranteed. ■

Proof of Proposition 2. In the (T, D)-plane, the saddle points match with the two areas:

$$\begin{aligned} -T - 1 &< D < T - 1 \\ T - 1 &< D < -T - 1 \end{aligned}$$

We knows, from Proposition 1, that $D < T - 1$. To show that the stationary state is a saddle point, one needs only to proof that $D > -T - 1$. Substituting formulas (28) and (29) into $D > -T - 1$, one gets the following condition:

$$2 + 2\left(\frac{\Delta_k + \rho}{\gamma} - \frac{\rho}{\gamma}\frac{\tau}{1-\tau}\left(\frac{y}{\gamma} + x\right)\right) + \frac{\rho}{\gamma}\left(\frac{\tau}{1-\tau}\frac{y}{\gamma} - \frac{\varepsilon_u \varepsilon_2}{\Delta_k + \rho}\frac{y}{x}\right) > 0$$

or, equivalently:

$$\gamma(1-\tau)\left(2x(\Delta_k + \rho)(\gamma + \Delta_k + \rho) - \rho y \varepsilon_u \varepsilon_2\right) - \rho \tau x (y + 2x\gamma)(\Delta_k + \rho) > 0 \tag{42}$$

Noticing that $\rho = (1-\tau)\varphi'$ and $\tau\varphi = \gamma - \Delta$, and dividing (42) by $1 - \tau$, yields to the rewritten condition:

$$2x\gamma(\Delta_k + \rho)(\gamma + \Delta_k + \rho) - \rho y \gamma \varepsilon_u \varepsilon_2 - \varepsilon_1(\gamma - \Delta)(\Delta_k + \rho)(y + 2x\gamma) > 0 \tag{43}$$

where $\varepsilon_1 \equiv x\varphi'/\varphi \in (0, 1)$. To show that (43) is satisfied, let us notice that $\Delta < \gamma$. We have immediately:

$$2x\gamma\left[(1+\varepsilon_1)\Delta + (1-\varepsilon_1)\gamma\right]/\varphi + \varepsilon_1\left[\tau x(\gamma - \Delta) + (1-\tau)(\gamma + \Delta)\right] > 0 \tag{44}$$

Multiplying both sides by φ, using the definition of ε_1 and reorganizing, yields to the following inequality:

$$2x\gamma\left[\gamma + \Delta + (1-\tau)\varphi'\right] - \varepsilon_1(\gamma - \Delta)\left[(1 - \tau - \tau x)\varphi + 2x\gamma\right] > 0 \tag{45}$$

Using (25), and Assumption 3 we get $y = (1 - \tau - \tau x)\varphi$; noticing that $\rho = (1 - \tau)\varphi'$, (45) can be rewritten:

$$2x\gamma(\gamma + \Delta + \rho) - \varepsilon_1(\gamma - \Delta)(y + 2x\gamma) > 0 \tag{46}$$

Multiplying (46) by $\Delta_k + \rho$ and adding the positive term $-\rho y\gamma\varepsilon_u\varepsilon_2$ one eventually gets (43) under assumption 3. ■

Proof of Proposition 3. Before maximization, we need to provide a steady state evaluation of the utility function: $(c_t, b_t, m_t, n_t) = (c_0, b_0, m_0, n_0)\gamma^t$, where γ is the regular growth factor common to all variables and impacting the defense services:

$$e_t \equiv e(m_t, n_t) = e\left(m_0\gamma^t, n_0\gamma^t\right) = e(m_0, n_0)\gamma^t = e_0\gamma^t$$

whose production function is assumed to be homogeneous of degree equal to one; denoting $e_0 \equiv e(m_0, n_0)$, one gets under Assumption (30):

$$\begin{aligned} W &= c_u \sum_{t=0}^{\infty} \beta^t \ln\left(c_0\gamma^t\right) + c_v \sum_{t=0}^{\infty} \beta^t \ln\left(b_0\gamma^t\right) + c_w \sum_{t=0}^{\infty} \beta^t \ln\left(e_0\gamma^t\right) \\ &= (c_u \ln c_0 + c_v \ln b_0 + c_w \ln e_0) \sum_{t=0}^{\infty} \beta^t + (c_u + c_v + c_w) \ln\gamma \sum_{t=0}^{\infty} \beta^t t \\ &= \frac{1}{1-\beta}\left(c_u \ln c_0 + c_v \ln b_0 + c_w \ln e_0 + \frac{\beta}{1-\beta}\ln\gamma\right) \end{aligned}$$

Uniqueness of the equilibrium (Proposition 1) requires c_0, b_0, e_0 being compatible with the stationary state γ, characterized by a regular growth. Definition (9) of economic policy implies at the first period: $(a_0, b_0, m_0, n_0) = (\sigma_a, \sigma_b, \sigma_m, \sigma_n) g_0$ and $e_0 = e(m_0, n_0) = e(\sigma_m g_0, \sigma_n g_0) = e(\sigma_m, \sigma_n) g_0$. From definition (18) one gets $c_0 = y g_0$. Remaining, that being at the endogenous growth rate steady state implies standing on the regular growth path, we have, under assumption (30):

$$\begin{aligned} W &= \frac{c_u \ln(y g_0) + c_v \ln(\sigma_b g_0) + c_w \ln(e(\sigma_m, \sigma_n) g_0)}{1-\beta} + \frac{\beta}{(1-\beta)^2}\ln\gamma \\ &= \frac{1}{1-\beta}\left(c_u \ln(y g_0) + c_v \ln(\sigma_b g_0) + c_w \ln\left[e(\sigma_m, \sigma_n) g_0\right] + \frac{\beta}{1-\beta}\ln\gamma\right) \\ &= \frac{1}{1-\beta}\left[c_u \ln y + c_v \ln\sigma_b + c_w \ln e(\sigma_m, \sigma_n) + \ln g_0 + \frac{\beta}{1-\beta}\ln\gamma\right] \end{aligned}$$

where $g_0 \equiv a_0 + b_0 + m_0 + n_0$ is an initial condition.

As β and g_0 do not belong to the set of variables used to maximize utility, the problem of maximizing W is equivalent to:

$$\max\left[c_u \ln y + c_v \ln\sigma_b + c_w \ln e(\sigma_m, \sigma_n) + \frac{\beta}{1-\beta}\ln\gamma\right] \tag{47}$$

Under Assumption 4, the public spending policy (9) implies:

$$\begin{aligned} f(\kappa, a, m) &\equiv F(\kappa, 1, a, m) = \theta \kappa^{\alpha} a^{\alpha_a} m^{\alpha_m} \\ \varphi(x_t) &= f(\kappa_t, a_t, m_t)/g_t = A(\kappa_t/g_t)^{\alpha}(a_t/g_t)^{\alpha_a}(m_t/g_t)^{\alpha_m} = \theta \sigma_a^{\alpha_a} \sigma_m^{\alpha_m} x_t^{\alpha} \\ \varphi'(x_t) &= \alpha \theta \sigma_a^{\alpha_a} \sigma_m^{\alpha_m} x_t^{\alpha-1} \end{aligned}$$

And still under Assumption 4: $\varepsilon_1 \equiv x\varphi'/\varphi = \alpha$.

As $\varepsilon_u = 1$, one gets from (24) the following implicit equation defining the stationary state x,

$$\Delta + \tau \theta \sigma_a^{\alpha_a} \sigma_m^{\alpha_m} x^{\alpha} = \beta \left[\Delta_k + (1-\tau) \alpha \theta \sigma_a^{\alpha_a} \sigma_m^{\alpha_m} x^{\alpha-1}\right]$$

Taking into account that $\tau\varphi = \gamma - \Delta$, equation (24) can be rewritten,

$$\gamma = \beta \left[\Delta_k + \frac{1-\tau}{\tau} \frac{1}{x} \frac{x\varphi'}{\varphi} (\gamma - \Delta)\right]$$

Substituting ε_1 into the above equation and solving to get γ gives the factor growth:

$$\gamma = \beta \frac{\Delta(1-\tau)\varepsilon_1 - \Delta_k \tau x}{(1-\tau)\beta\varepsilon_1 - \tau x}$$

From (25) and the above definition of γ, problem (47) becomes:

$$\begin{aligned} &\max c_v \ln \sigma_b + c_w \ln e(\sigma_m, \sigma_n) + c_u \ln\left[(1-\tau-\tau x)\varphi(x) + (\Delta_k - \Delta)x\right] \\ &+ \frac{\beta}{1-\beta} \ln \left[\beta \frac{\Delta(1-\tau)\varepsilon_1 - \Delta_k \tau x}{(1-\tau)\beta\varepsilon_1 - \tau x}\right] \end{aligned} \tag{48}$$

with

$$\frac{\Delta\varepsilon_1(1-\tau) - \Delta_k \tau x}{\beta\varepsilon_1(1-\tau) - \tau x} = \Delta_k + (1-\tau)\varphi'(x)$$

and still, (10) and (12).

Noticing that $\tau\varphi = \gamma - \Delta$ implies

$$\varphi(x) = \frac{(\Delta - \beta\Delta_k)x}{\beta\varepsilon_1(1-\tau) - \tau x} \tag{49}$$

and then substituting $\varphi(x)$ into (48) yields to the following program:

$$\begin{aligned} &\max c_u \ln x + c_v \ln \sigma_b + c_w \ln e(\sigma_m, \sigma_n) \\ &+ c_u \ln \left[(\Delta - \beta\Delta_k) \frac{1-\tau-\tau x}{\beta\varepsilon_1(1-\tau) - \tau x} + \Delta_k - \Delta\right] \\ &+ \frac{\beta}{1-\beta} \ln \left[\beta \frac{\Delta\varepsilon_1(1-\tau) - \Delta_k \tau x}{\beta\varepsilon_1(1-\tau) - \tau x}\right] \end{aligned}$$

which can be rewritten under Assumption 5,

$$\begin{aligned} &\max c_u \ln x + c_v \ln \sigma_b + c_w \ln e(\sigma_m, \sigma_n) + c_u \ln\left[(1-\beta)\Delta\right] \\ &+ \frac{\beta}{1-\beta} \ln(\beta\Delta) + c_u \ln \frac{1-\tau-\tau x}{\beta\varepsilon_1(1-\tau) - \tau x} + \frac{\beta}{1-\beta} \ln \frac{\varepsilon_1(1-\tau) - \tau x}{\beta\varepsilon_1(1-\tau) - \tau x} \end{aligned}$$

Noticing that $\sigma_b = 1 - \sigma_a - \sigma_m - \sigma_n$ - restriction (10) - and that β and Δ do not belong to the set of endogenous variables for the maximizing problem, one gets the new program:

$$\max c_u \ln x + c_w \ln e(\sigma_m, \sigma_n) + c_v \ln(1 - \sigma_a - \sigma_m - \sigma_n) + c_u \ln(1 - \tau - \tau x)$$
$$+\frac{\beta}{1-\beta} \ln[\varepsilon_1(1-\tau) - \tau x] - \left(c_u + \frac{\beta}{1-\beta}\right) \ln[\beta\varepsilon_1(1-\tau) - \tau x]$$

Under assumption 4, the implicit equation (49) becomes,

$$\theta \sigma_a^{\alpha_a} \sigma_m^{\alpha_m} x^{\alpha} = \frac{(1-\beta)\Delta x}{\beta\varepsilon_1(1-\tau) - \tau x}$$

This equation defines locally a function $x = x(\tau, \sigma_a, \sigma_m)$. We thus need to maximize, with respect to variables $(\tau, \sigma_a, \sigma_m, \sigma_n)$, an increasing function $\tilde{W}$ of the initial welfare function W:

$$\begin{aligned}\tilde{W} \equiv\ & c_u \ln x(\tau, \sigma_a, \sigma_m) + c_w \ln e(\sigma_m, \sigma_n) + c_v \ln(1 - \sigma_a - \sigma_m - \sigma_n) \\ & + c_u \ln(1 - \tau - \tau x(\tau, \sigma_a, \sigma_m)) + \frac{\beta}{1-\beta} \ln[(1-\tau)\varepsilon_1 - \tau x(\tau, \sigma_a, \sigma_m)] \\ & - \left(c_u + \frac{\beta}{1-\beta}\right) \ln[(1-\tau)\beta\varepsilon_1 - \tau x(\tau, \sigma_a, \sigma_m)]\end{aligned}$$

To simplify the writing, the following notations will denote partial derivatives:

$$x_\tau \equiv \frac{\partial x}{\partial \tau}, x_a \equiv \frac{\partial x}{\partial \sigma_a}, x_m \equiv \frac{\partial x}{\partial \sigma_m}, e_m \equiv \frac{\partial e}{\partial \sigma_m}, e_n \equiv \frac{\partial e}{\partial \sigma_n}$$

All the restrictions being already include in the program, one simply needs to solve an unconstrained optimization problem, by setting the gradient equal to zero:[12]

$$\frac{\partial \tilde{W}}{\partial \tau} = \frac{\partial \tilde{W}}{\partial \sigma_a} = \frac{\partial \tilde{W}}{\partial \sigma_m} = \frac{\partial \tilde{W}}{\partial \sigma_n} = 0$$

One gets respectively:

$$c_u \frac{x_\tau}{x} - c_u \frac{1 + x + \tau x_\tau}{1 - \tau - \tau x} - \frac{\beta}{1-\beta} \frac{x + \tau x_\tau + \varepsilon_1}{\varepsilon_1(1-\tau) - \tau x} + \left(c_u + \frac{\beta}{1-\beta}\right) \frac{x + \tau x_\tau + \beta\varepsilon_1}{\beta\varepsilon_1(1-\tau) - \tau x} = 0 \tag{50}$$

$$c_u \frac{x_a}{x} - c_u \frac{\tau x_a}{1 - \tau - \tau x} - \frac{\beta}{1-\beta} \frac{\tau x_a}{\varepsilon_1(1-\tau) - \tau x} + \left(c_u + \frac{\beta}{1-\beta}\right) \frac{\tau x_a}{\beta\varepsilon_1(1-\tau) - \tau x} = \frac{c_v}{1 - \sigma_a - \sigma_m - \sigma_n} \tag{51}$$

$$c_u \frac{x_m}{x} - c_u \frac{\tau x_m}{1 - \tau - \tau x} - \frac{\beta}{1-\beta} \frac{\tau x_m}{\varepsilon_1(1-\tau) - \tau x} + \left(c_u + \frac{\beta}{1-\beta}\right) \frac{\tau x_m}{\beta\varepsilon_1(1-\tau) - \tau x} = \frac{c_v}{1 - \sigma_a - \sigma_m - \sigma_n} - c_w \frac{e_m}{e} \tag{52}$$

[12] Concerning the second order conditions, the concavity of the initial problem guarantees that the Hessian matrix of $\tilde{W}$ with respect to $(\tau, \sigma_a, \sigma_m, \sigma_n)$ is negative semi-definite.

and

$$c_w \frac{e_n}{e} = \frac{c_v}{1 - \sigma_a - \sigma_m - \sigma_n} \tag{53}$$

Using (51), (52) and (53), we have

$$\begin{aligned} & c_u \frac{1}{x} - c_u \frac{\tau}{1-\tau-\tau x} - \frac{\beta}{1-\beta} \frac{\tau}{\varepsilon_1 (1-\tau) - \tau x} + \left(c_u + \frac{\beta}{1-\beta} \right) \frac{\tau}{\beta \varepsilon_1 (1-\tau) - \tau x} \\ = & \frac{c_w}{x_m} \frac{e_n}{e} - \frac{c_w}{x_m} \frac{e_m}{e} = \frac{c_w}{x_a} \frac{e_n}{e} \end{aligned} \tag{54}$$

and

$$\frac{x_m}{x_a} = 1 - \frac{e_m}{e_n} \tag{55}$$

Partial derivatives x_τ, x_a, x_m are easily obtained by differentiating totally

$$\varphi (x, \sigma_a, \sigma_m) = \frac{(1-\beta) \Delta x}{\beta \varepsilon_1 (1-\tau) - \tau x}$$

and remembering that $\varphi (x, \sigma_a, \sigma_m) = \theta \sigma_a^{\alpha_a} \sigma_m^{\alpha_m} x^{\alpha}$. Noticing that $\varepsilon_1 = \alpha$, $\sigma_a \varphi_a / \varphi = \alpha_a$, $\sigma_m \varphi_m / \varphi = \alpha_m$, one gets:

$$x_\tau \equiv \frac{\partial x}{\partial \tau} = -\frac{x}{\varepsilon_1} \frac{\beta \varepsilon_1 + x}{\beta (1-\varepsilon_1)(1-\tau) + \tau x} \tag{56}$$

$$x_a \equiv \frac{\partial x}{\partial \sigma_a} = \frac{\alpha_a}{\sigma_a} \frac{x}{\varepsilon_1} \frac{1}{\beta (1-\varepsilon_1)(1-\tau) + \tau x} \tag{57}$$

$$x_m \equiv \frac{\partial x}{\partial \sigma_m} = \frac{\alpha_m}{\sigma_m} \frac{x}{\varepsilon_1} \frac{1}{\beta (1-\varepsilon_1)(1-\tau) + \tau x} \tag{58}$$

Equation (55) gives

$$\frac{\alpha_m \sigma_a}{\alpha_a \sigma_m} = 1 - \frac{e_m}{e_n} \tag{59}$$

and under Assumption 4, it is easy to notice that:

$$e_m / e = \beta_m / \sigma_m$$

$$e_n / e = \beta_n / \sigma_n \tag{60}$$

$$e_m / e_n = \beta_m \sigma_n / (\beta_n \sigma_m) \tag{61}$$

From system (53-59), using (60) and (61), one gets σ_m and σ_n as linear functions of σ_a (equations (34) and (35)). Substituting (56) into (50) and solving, gives τ as a function of x (equation (32)); we keep the solution $\tau(x) \in [0,1]$. Substituting (57), (34) and (35) into (51), one gets σ_a as a function of x and τ (equation (33)). One eventually gets the optimal policy by solving equation (31) for x, with τ given by (32), whereas σ_m and σ_a are given respectively by (34) and (33). Once x^* has been determined, we compute τ^* from (32), σ_a^* from (33) and yet σ_m^*, σ_n^* and σ_b^* from (34), (35) and (10), respectively. ■

9 References

Aizenman J. & Glick R. [2003], Military Expenditure, Threats and Growth, NBER Working Papers 9618, National Bureau of Economic Research Inc..

Aghion P. & Howitt P. [1997], *Endogenous Growth Theory*, MIT Press.

Barro R. J. [1990], Government Spending in a Simple Model of Endogenous Growth, *Journal of Political Economy* 98, 103-25.

Barro R. J. & Sala-i-Martín X. [1995], *Economic Growth*, MIT Press.

Benoit E. [1973], *Defense and Economic Growth in Developing Countries*, Lexington, MA, D.C., Heath.

Benoit, E. [1978], Growth and Defense in Developing Countries, *Economic Development and Cultural Change* 26, 271-280.

Biswas B. & Ram R. [1986], Military Expenditures and Economic Growth in Less Developed Countries: an Augmented Model and Further Evidence, *Economic Development and Cultural Change* 34, 361-372.

Brito D. [1972], A Dynamic Model of an Armaments Race, *International Economic Review* 13, 359-375.

Brumm H.J. [1997], Military Spending, Government Disarray, and Economic Growth: a Cross-Country Empirical Analysis, *Journal of Macroeconomics* 19, 827- 838.

Chang W.Y., Tsai H.F. & Lai C.C. [1996], Effects of anticipated foreign military threats on arms accumulation, *Southern Economic Journal* 63, 507-514.

Dakurah H., Davies S. & Sampath R. [2001], Defense Spending and Economic Growth in Developing Countries: A Causality Analysis, *Journal of Policy Modelling* 23, 651-658.

Deger S. [1986], Economic development and defense expenditure, *Economic Development and Cultural Change* 35, 179-196.

Deger S. & Smith R. [1983], Military Expenditure and Development: The Economic Linkages, *IDS Bulletin* 16, 49-54.

Deger S. & Sen S. [1983], Military Expenditure, Spin-Off and Economic Development, *Journal of Development Economics* 13, 67-83.

Deger S. & Sen S. [1984], Optimal Control and Differential Game Models of Military Expenditures in Less Developed Countries, *Journal of Economic Dynamics and Control* 7, 153-169.

Dunne J.P., Smith R. & Willenbockel D. [2004], Models of Military Expenditure and Growth: A Critical Review, *Defence and Peace Economics* 16, 449-461.

Faini R., Annez P. & Taylor L. [1984], Defense Spending, Economic Structure, and Growth: Evidence among Countries and over Time, *Economic Development and Cultural Change* 32, 487-498.

Huand C. & Mintz A. [1991], Defense Expenditures and Economic Growth: the Externality Effect, *Defense Economics* 3, 35-40.

Lipow J. [1990], Defense, Growth, and Disarmament: A Further Look, *Jerusalem Journal of International Relations* 12, 49-59.

Mankiw N. G., Romer D. & Weil D. N. [1992], A Contribution to the Empirics of EconomicGrowth, *Quarterly Journal of Economics* 107, 407-437.

Macnair E. S., Murdoch J.C., Pi C.R. & Sandler T. [1995], Growth and Defense: Pooled Estimates for Three NATO Alliance, 1951-1988. *Southern Economic Journal* 61, 846-860.

McNamara R. [1991], The Post-Cold War World: Implications for Military Expenditure in the Developing Countries, World Bank Annual Conference on Development Economics, Washington, DC, the World Bank, 95-144.

Murdoch J.C., Pi C.R. & Sandler T. [1997], The Impact of Defense and Nondefense Public Spending on Growth in Asia and Latin America, *Defense and Peace Economics* 8, 205-224.

Ram R. [1995], Defense Expenditure and Economic Growth, in Hartley, K., Sandler, T. (Eds.), *Handbook of Defense Economics*, Elsevier, Amsterdam, 251-273.

Sandler T. & Hartley K. [1995], *The Economics of Defense*, Cambridge Univ. Press, Cambridge.

Shieh J., Lai C.C. & Chang W.Y. [2002], The Impact of Military Burden on Long-Run Growth and Welfare, *Journal of Development Economics* 68, 443-454.

Van der Ploeg F. & de Zeeuw A. [1990], Perfect Equilibrium in a Model of Competitive Arms Accumulation, *International Economic Review* 31, 131-146.

Zou H. [1995], A Dynamic Model of Capital and Arms Accumulation, *Journal of Economic Dynamics and Control* 19, 371-393.

10 Procedure

10.1 Marginal policy

We have

$$\begin{aligned}
\frac{d\gamma}{d\sigma_m} &= (\gamma-\Delta)\frac{\alpha_m}{\sigma_m}\left(1+\frac{1}{\beta(1-\varepsilon_1)(1-\tau)+\tau\left(\frac{\gamma-\Delta}{\tau\theta\sigma_a^{\alpha_a}\sigma_m^{\alpha_m}}\right)^{1/\alpha}}\right)\\
\frac{d\gamma}{d\sigma_m} &= 0.200\,082\,416\,2\\
\gamma+\frac{d\gamma}{d\sigma_m}\Delta\sigma_m &= 1.\,003\,729\,089+0.200\,082\,416\,2*(0.004-0.002)=1.\,004\,129\,254\\
1.\,004\,129\,254^4-1 &= 0.016\,619\,602
\end{aligned}$$

10.2 Optimal policy

$\beta = 0.985$
$\Delta = 1 - 0.02$
$\varepsilon_1 = \frac{75}{100}$
$\alpha = \varepsilon_1$
$\alpha_a = \frac{24.5}{100}$
$\alpha_m = \frac{0.5}{100}$
$\gamma = (1.015)^{1/4}$
$\tau = 0.4$
$\sigma_a = 0.179$
$\sigma_b = 0.761$
$\sigma_m = 0.002$
$\sigma_n = 0.058$
$\theta = 0.123\,982\,284\,6$

$$\frac{d\gamma}{d\sigma_m} = (\gamma-\Delta)\frac{\alpha_m}{\sigma_m}\left(1+\frac{1}{\beta(1-\varepsilon_1)(1-\tau)+\tau\left(\frac{\gamma-\Delta}{\tau\theta\sigma_a^{\alpha_a}\sigma_m^{\alpha_m}}\right)^{1/\alpha}}\right) = 0.200\,082\,416\,2$$

$c_u = \frac{9}{20}$
$c_v = \frac{9}{20}$
$c_w = \frac{2}{20}$
$\beta_m = 1/4$
$\beta_n = 3/4$
$\tau = \left(1+A-B-\sqrt{(1+A-B)^2-4A}\right)/2$
$A = \frac{\varepsilon_1(\varepsilon_1 c_u(\beta+x)+(1-\varepsilon_1)(\varepsilon_1 c_u-1)\beta x)}{c_u(\varepsilon_1+x)(\beta\varepsilon_1+2\beta\varepsilon_1 x+x^2)-\beta\varepsilon_1 x(1+x)}$
$B = \frac{x^2((c_u(1+\varepsilon_1)-\varepsilon_1)\beta\varepsilon_1+c_u(x-\varepsilon_1))}{c_u(\varepsilon_1+x)(\beta\varepsilon_1+2\beta\varepsilon_1 x+x^2)-\beta\varepsilon_1 x(1+x)}$

$$\sigma_a = \left(1 + \frac{\alpha_m}{\alpha_a} + \frac{\varepsilon_1}{\alpha_a}\frac{(c_v+(\beta_n+\beta_m)c_w)(\beta(1-\varepsilon_1)(1-\tau)+\tau x)}{c_u + c_u\frac{\tau x}{\tau x-(1-\tau)} + \frac{\beta}{1-\beta}\frac{\tau x}{\tau x-(1-\tau)\varepsilon_1} - \left(c_u+\frac{\beta}{1-\beta}\right)\frac{\tau x}{\tau x-(1-\tau)\beta\varepsilon_1}}\right)^{-1}$$

$$s_m = \frac{\alpha_m(c_v+\beta_n c_w)-\alpha_a\beta_m c_w}{\alpha_a(c_v+\beta_n c_w+\beta_m c_w)}\sigma_a + \frac{\alpha_a\beta_m c_w}{\alpha_a(c_v+\beta_n c_w+\beta_m c_w)}$$

$$x = 3.765$$

$$\theta\sigma_a^{\alpha_a}s_m^{\alpha_m}x^{\alpha}, \frac{x\Delta(1-\beta)}{\beta\varepsilon_1(1-\tau)-\tau x}$$

$$\tau = \left(1+A-B-\sqrt{(1+A-B)^2-4A}\right)/2 = 0.125\,535\,653\,9$$

$$\sigma_a = \left(1 + \frac{\alpha_m}{\alpha_a} + \frac{\varepsilon_1}{\alpha_a}\frac{(c_v+(\beta_n+\beta_m)c_w)(\beta(1-\varepsilon_1)(1-\tau)+\tau x)}{c_u + c_u\frac{\tau x}{\tau x-(1-\tau)} + \frac{\beta}{1-\beta}\frac{\tau x}{\tau x-(1-\tau)\varepsilon_1} - \left(c_u+\frac{\beta}{1-\beta}\right)\frac{\tau x}{\tau x-(1-\tau)\beta\varepsilon_1}}\right)^{-1} = 0.886\,481\,848$$

$$\sigma_b = 1-\sigma_a-\sigma_m-\sigma_n = 0.078\,076\,378\,6$$

$$= \left(1 - \left(\left(1 + \frac{\alpha_m}{\alpha_a} + \frac{\varepsilon_1}{\alpha_a}\frac{(c_v+(\beta_n+\beta_m)c_w)(\beta(1-\varepsilon_1)(1-\tau)+\tau x)}{c_u + c_u\frac{\tau x}{\tau x-(1-\tau)} + \frac{\beta}{1-\beta}\frac{\tau x}{\tau x-(1-\tau)\varepsilon_1} - \left(c_u+\frac{\beta}{1-\beta}\right)\frac{\tau x}{\tau x-(1-\tau)\beta\varepsilon_1}}\right)^{-1}\right) - \Big(\right.$$

$$\sigma_m = \frac{\alpha_m(c_v+\beta_n c_w)-\alpha_a\beta_m c_w}{\alpha_a(c_v+\beta_n c_w+\beta_m c_w)}\sigma_a + \frac{\alpha_a\beta_m c_w}{\alpha_a(c_v+\beta_n c_w+\beta_m c_w)} = 2.242\,904\,288\times10^{-2}$$

$$\sigma_n = -\frac{(\alpha_a+\alpha_m)\beta_n c_w}{\alpha_a(c_v+\beta_n c_w+\beta_m c_w)}\sigma_a + \frac{\beta_n c_w}{c_v+\beta_n c_w+\beta_m c_w} = 0.013\,012\,729\,8$$

$$\gamma = \Delta + \tau\theta\sigma_a^{\alpha_a}s_m^{\alpha_m}x^{\alpha} = 1.020\,075\,932$$